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Bathtub and Related Failure Rate Characterizations

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Sufficient conditions are obtained which provide that a lifetime density has a bathtub shaped failure rate. Analogous conditions handle increasing, decreasing, and upside-down bathtub shaped failure rates. Application of these results to exponential families of densities is particularly straightforward and effective. Examples are furnished which introduce new bathtub models and illustrate the use of the general results for existing models. Examples involving mixtures are considered. Maximum likelihood estimation for one of the bathtub models is described.

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SUMMARY

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1. INTRODUCTION

The probability distribution of the time-to-failure of an item can be characterized by the failure rate, $r(t) = f(t)/R(t)$, where $f(t)$ denotes the density function and $R(t)$ the reliability, or probability of failure after time t. The failure rate has a probabilistic interpretation: $r(t)dt$ represents the probability that an item of age t will fail in the interval $(t, t + dt)$.

Many parametric lifetime models, such as the gamma, Weibull, and truncated normal distributions, have monotone failure rate. If $r(t)$ increases monotonically over time, the distribution is said to have increasing failure rate (IFR). If $r(t)$ decreases monotonically, we have decreasing failure rate (DFR). The IFR property is characteristic of devices which consistently deteriorate with age, whereas the DFR property is characteristic of devices which consistently improve with age. Many physical phenomena exhibit failure rates which are non-monotonic. A common description, which is appropriate for modeling

human lifetimes, shows three phases: an initial phase where the failure rate decreases, followed by a middle phase where the failure rate is essentially constant, concluded by a final phase where the failure rate increases. For humans, the first (infant mortality) phase shows deaths due to hereditary defects, whose impact dimishes with time. The middle (chance failure) phase shows deaths due typically to sudden jolts, such as accidents. The final (wear-out) phase shows death resulting from the natural accumulation of negative effects. Such failure rates are usually termed bathtub (BT) shaped. The logical counterpart to BT failure rate is the three phase situation where the failure rate initially increases, then becomes essentially constant, and ultimately decreases. This failure rate function, which we will term upside-down bathtub (UBT) shaped, can be found in accelerated life testing, where the items tested are subjected to abnormally high stress levels. The lognormal and inverse Gaussian lifetime models, as we shall show, have UBT shaped failure rates. For the sake of brevity in this paper, we shall frequently say that the failure time model, distribution, or f is BT, IFR, UBT, or DFR, when we mean more precisely that the associated failure rate function is BT shaped, increasing, etc.

Although bathtub shaped lifetime models are of great practical value, few have been suggested in the literature. See Lieberman (1969). One reason for this scarcity undoubtedly has been the difficulty in ascertaining whether a given $r(t)$ is bathtub shaped. So motivated, we obtain in Section 2 a general result which furnishes sufficient conditions that a distribution has a BT shaped failure rate function. Dual results handle the UBT, IFR, and DFR situations. For lifetime distributions of the exponential family type, or mixtures of the same, these

conditions have a simple, easy to test form. This ease of application is demonstrated in Section 3, where new BT lifetime models are proposed. In Section 4, statistical inference is considered for the new models.

The applicability of the general results to mixtures is useful. In certain situations, it is reasonable to assume that a proportion p of items in a population come from one distribution (say IFR, characterized by wear-out failures), and the remaining proportion, $q = 1 - p$, of items come from another distribution (say DFR, characterized by catastrophic failures). An example with electron tubes is considered by Kao (1959). We see in Section 3 that for certain gamma mixtures the combined population has a BT shaped failure rate function.

We shall assume throughout this paper that the failure time distribution has support $(0,M)$, i.e., $\{t:f(t) > 0\} = (0,M)$. The constant M may be, and indeed in most applications will be, ∞. Because of further assumptions placed on the density function in Section 2, the failure rate function $r(t)$ will be continuous and twice differentiable for all $t \in (0,M)$. With exception of the exponential distribution, where $r'(t) = 0$ for all $t > 0$, resulting failure rate functions will exhibit properties of strict monotonicity. For brevity, we denote by (I) the strict IFR situation, "$r'(t) > 0$ for all $t \in (0,M)$," and denote by (D) the strict DFR situation, "$r'(t) < 0$ for all $t \in (0,M)$." Similarly, we denote by (B) the special BT situation, "for some $t^* \in (0,M)$, $r'(t) < 0$ for all $t \in (0,t^*)$, $r'(t^*) = 0$, and $r'(t) > 0$ for all $t \in (t^*,M)$." Finally, we denote by (U) the special UBT situation, "for some $t^* \in (0,M)$, $r'(t) > 0$ for all $t \in (0,t^*)$, $r'(t^*) = 0$, and $r'(t) < 0$ for all $t \in (t^*,M)$."

2. GENERAL RESULTS

In this section we shall obtain general results which supply sufficient conditions to characterize a given failure time distribution as being either BT, IFR, UBT, or DFR. We assume throughout that the failure time distribution is absolutely continuous with support $(0,M)$, where the constant M may be ∞. We assume the density $f(t)$ is continous and twice differentiable on $(0,M)$.* We define $g(t)$ as the reciprocal of the failure rate,

$$g(t) = 1/r(t) = R(t)/f(t). \tag{2.1}$$

It follows that $g(t)$ is positive valued, continuous, and twice differentiable on $(0,M)$. In fact, we have

$$g'(t) = g(t)\eta(t) - 1, \tag{2.2}$$

where $\eta(t)$ is defined by

$$\eta(t) = - f'(t)/f(t). \tag{2.3}$$

THEOREM. Suppose $g'(t)$ can be expressed as

$$g'(t) = \int_t^M s_t(y)[\eta(t) - \eta(y)]dy, \tag{2.4}$$

for some function $s_t(y)$ which is positive valued for all t and y in $(0,M)$.

(a) If $\eta'(t) > 0$ for all $t \in (0,M)$, then (I).

(b) If $\eta'(t) < 0$ for all $t \in (0,M)$, then (D).

*All differentiation mentioned and performed in this section will refer to partial differentiation with respect to the time argument t; i.e., all parameter values are held fixed. Such derivatives will be denoted by ' and ".

(c) Suppose there exists $t_0 \in (0,M)$ such that

$$\eta'(t) < 0 \text{ for all } t \in (0,t_0),\ \eta'(t_0) = 0,$$
$$\text{and} \quad \eta'(t) > 0 \text{ for all } t \in (t_0,M). \tag{2.5}$$

(i) If there exists $y_0 \in (0,M)$ such that $g'(y_0) = 0$, then (B).

(ii) If there does not exist $y_0 \in (0,M)$ such that $g'(y_0)=0$, then (I).

(d) Suppose there exists $t_0 \in (0,M)$ such that

$$\eta'(t) > 0 \text{ for all } t \in (0,t_0),\ \eta'(t_0) = 0,$$
$$\text{and} \quad \eta'(t) < 0 \text{ for all } t \in (t_0,M). \tag{2.6}$$

(i) If there exists $y_0 \in (0,M)$ such that $g'(y_0) = 0$, then (U).

(ii) If there does not exist $y_0 \in (0,M)$ such that $g'(y_0)=0$, then (D).

PROOF. (a) The assumption that $\eta'(t) > 0$ for all $t \in (0,M)$ implies, from (2.4), that $g'(t) < 0$ for all $t \in (0,M)$, which, from (2.1), implies (I).

(b) Here it follows that $g'(t) > 0$ for all $t \in (0,M)$, which implies (D).

(c,i) Claim $g''(y_0) < 0$. Since $g'(y_0) = 0$, it follows from differentiation of (2.2) that $g''(y_0) = g(y_0)\eta'(y_0)$. Therefore $g''(y_0) < 0 \Leftrightarrow \eta'(y_0) < 0 \Leftrightarrow y_0 < t_0$. Suppose $y_0 \geq t_0$. From (2.4) it is apparent that $g'(t) < 0$ for all $t \in [t_0,M)$. Therefore $g'(y_0) < 0$, which is a contradiction. Hence $y_0 < t_0$ and $g''(y_0) < 0$. It is clear that there is only one root in $(0,M)$ to $g'(y) = 0$, namely $y = y_0$, and g attains a maximum at this point. This implies (B), with $t^* = y_0$.

(c,ii) Here either $g'(t) > 0$ for all $t \in (0,M)$ or $g'(t) < 0$ for all $t \in (0,M)$. From (2.4) we have that $g'(t) < 0$ for all $t \in [t_0,M)$. Therefore $g'(t) < 0$ for all $t \in (0,M)$, and (I) holds.

(d) The proof is analogous to that of (c) and will be omitted.

The Theorem is readily applicable to exponential families of densities. Suppose f can be expressed as

$$f(t) = C(\underset{\sim}{\theta}) \exp\left\{\Sigma_{i=1}^{k} \theta_i U_i(t)\right\}, \quad 0 < t < M, \tag{2.7}$$

where for each i, $U_i(t)$ is twice differentiable on $(0,M)$. Then from (2.3),

$$\eta(t) = -\Sigma_{i=1}^{k} \theta_i U_i'(t), \text{ and } \eta'(t) = -\Sigma_{i=1}^{k} \theta_i U_i''(t). \tag{2.8}$$

The condition (2.4) of the theorem always holds in this situation. For, consider

$g(t) = \int_t^M [f(y)/f(t)]dy = \int_0^M [f(w+t)/f(t)]dw = \int_0^M \exp\left\{\Sigma_{i=1}^{k} \theta_i [U_i(w+t) - U_i(t)]\right\} dw.$

Therefore $g'(t) = \int_0^M \exp\left\{\Sigma_{i=1}^{k} \theta_i [U_i(w+t) - U_i(t)]\right\} \Sigma_{i=1}^{k} \theta_i [U_i'(w+t) - U_i'(t)]dw$

$$= \int_t^M \exp\left\{\Sigma_{i=1}^{k} \theta_i [U_i(y) - U_i(t)]\right\} [\eta(t) - \eta(y)]dy,$$

and (2.4) holds, with $s_t(y) = \exp\left\{\Sigma_{i=1}^{k} \theta_i [U_i(y) - U_i(t)]\right\} > 0$. We have established the following result.

COROLLARY 1. Suppose f has the exponential family form (2.7). Then the assertions (a) through (d) of the Theorem hold, where η' is given by (2.8).

It should be noted that the form (2.7) is quite general. Typically the vector $\underset{\sim}{\theta} = (\theta_1,\ldots,\theta_k)$ is a vector of parameters. In using Corollary 1, a density of the ostensibly more general form $f(t)=C(\underset{\sim}{\theta})h(t)\exp\{\Sigma_{i=1}^{k}\theta_i U_i(t)\}$ should be written as $f(t)=C(\underset{\sim}{\theta}^*)\exp\{\Sigma_{i=1}^{k*}\theta_i^* U_i^*(t)\}$, where $k^* = k+1$, $\underset{\sim}{\theta}^* = (\theta_1,\ldots,\theta_k,1)$, and $U_i^*(t) = U_i(t)$, $i=1,\ldots,k$, with $U_{k+1}^*(t)=\log h(t)$.

The Theorem is applicable also to certain exponential family mixtures. Suppose f can be expressed as

$$f(t) = p\, f_1(t) + q\, f_2(t), \tag{2.9}$$

where $0 < p < 1$, $q = 1-p$, and

$$f_j(t) = C_j(\underset{\sim}{\theta})\exp\{\Sigma_{i=1}^{k}\theta_{ij}U_i(t)\}, \quad 0<t<M, \tag{2.10}$$

with each $U_i(t)$ being twice differentiable on $(0,M)$. We note that $\eta(t) = f'(t)/f(t) = -[f_1'(t) + c\, f_2'(t)]/[f_1(t) + c\, f_2(t)]$, where c is defined by $c = q/p$. Since $f_1(t) + c\, f_2(t) = c\, C_2(\underset{\sim}{\theta})\exp\{\Sigma_{i=1}^{k}\theta_{i2}U_i(t)\}\xi(t)$, where $\xi(t)$ is defined by

$$\xi(t) = 1 + [C_1(\underset{\sim}{\theta})/c\, C_2(\underset{\sim}{\theta})]\exp\{\Sigma_{i=1}^{k}(\theta_{i1} - \theta_{i2})U_i(t)\}, \tag{2.11}$$

it follows that $\eta(t) = -[\xi'(t)/\xi(t) + \Sigma_{i=1}^{k}\theta_{i2}U_i'(t)]$. For computational purposes it is useful to note that

$$\eta(t) = -\{\Sigma_{i=1}^{k}\theta_{i1}U_i'(t) + [\Sigma_{i=1}^{k}(\theta_{i2}-\theta_{i1})U_i'(t)]/\xi(t)\}. \tag{2.12}$$

We establish the applicability of the Theorem by showing that the condition (2.4) always holds here. Since $g(t) = \int_0^M [f(w+t)/f(t)]dw$, we have $g'(t) = \int_0^M [f(w+t)/f(t)]'dw$. But $f(w+t)/f(t) =$ $[f_1(w+t)+c\, f_2(w+t)]/[f_1(t)+c\, f_2(t)] = [\xi(w+t)/\xi(t)]\exp\{\Sigma_{i=1}^{k}\theta_{i2}[U_i(w+t)-U_i(t)]\}$. Therefore $[f(w+t)/f(t)]' = \exp\{\Sigma_{i=1}^{k}\theta_{i2}[U_i(w+t)-U_i(t)]\}\{\xi'(w+t)/\xi(t)$ $- \xi(w+t)\xi'(t)/[\xi(t)]^2 + [\xi(w+t)/\xi(t)]\Sigma_{i=1}^{k}\theta_{i2}[U_i'(w+t)-U_i'(t)]\}$. Consequently,

$$g'(t)=\int_t^M \exp\{\Sigma_{i=1}^k \theta_{i2}[U_i(y)-U_i(t)]\}[\xi(y)/\xi(t)]\{\xi'(y)/\xi(y)-\xi'(t)/\xi(t)$$

$$+\Sigma_{i=1}^k \theta_{i2}[U_i'(y)-U_i'(t)]\}dy = \int_t^M s_t(y)[\eta(t)-\eta(y)]dy, \text{ where}$$

$s_t(y)=\exp\{\Sigma_{i=1}^k \theta_{i2}[U_i(y)-U_i(t)]\}[\xi(y)/\xi(t)] > 0$ for all t and y in $(0,M)$. We therefore have the following result.

COROLLARY 2. Suppose f has the mixture of exponential families form described by (2.9) and (2.10). Then the assertions (a) through (d) of the Theorem hold, were η' can be obtained from (2.12) and (2.11).

In concluding this section, we present an approach which can speed implementation of the Theorem and Corollaries in situations (c) or (d), where (2.5) or (2.6) hold, respectively. Suppose (2.5) holds. The crucial issue is whether (i) or not (ii) a number $y_0 \in (0,M)$ exists satisfying $g'(y_0) = 0$. From the proof of (c), it is clear that either $g(t)$ has a single extremum in $(0,M)$, namely a maximum (case (i), (B)), or $g(t)$ is strictly monotone decreasing (case (ii), (I)). The crucial issue can therefore be stated in the typically more tractable form of whether (i) g is increasing initially (i.e., in a neighborhood of 0) or (ii) g is decreasing initially. To appreciate the usefulness of this representation, consider the behavior of g near 0. Suppose $\lim_{t\downarrow 0} g(t)$ exists and equals 0. (This is equivalent to $\lim_{t\downarrow 0} f(t) = \infty$.) Then since $g(t) > 0$ for all $t \in (0,M)$, we conclude that g must be increasing initially, (i) holds, and f is BT. Similarly, $\lim_{t\downarrow 0} f(t) = 0$, or $\lim_{t\downarrow 0} g(t) = \infty$, implies (ii) holds, and f is IFR. If $f(t)$ does not tend to 0 or ∞, it is then expedient to consider the behavior of

$g(t)\eta(t)$ near 0. Suppose $\delta = \lim_{t\downarrow 0} g(t)\eta(t)$ exists, possibly equal to $+\infty$ or $-\infty$. From (2.2), we note that $\delta > 1$ implies $g'(t) > 0$ in a neighborhood of 0, so that (i) holds, and f is BT. On the other hand, $\delta < 1$ implies (ii) holds, and f is IFR. If $\delta=1$, higher derivatives of g near 0 may be investigated to determine whether g is initially increasing or decreasing. The situation of (2.6) is analogous to that of (2.5), with the issue being whether g is decreasing initially (case (i), (U)) or increasing intially (case (ii), (D)). The following summarizes our modified approach.

LEMMA. Suppose (2.5) or (2.6) holds in the Theorem or in Corollary 1 or 2.

(a) Suppose $\varepsilon = \lim_{t\downarrow 0} f(t)$ exists, possibly equal to 0 or ∞.

(i) If $\varepsilon = \infty$ and (2.5) holds, then (B).

(ii) If $\varepsilon = 0$ and (2.5) holds, then (I).

(iii) If $\varepsilon = 0$ and (2.6) holds, then (U).

(iv) If $\varepsilon = \infty$ and (2.6) holds, then (D).

(b) Suppose $\delta = \lim_{t\downarrow 0} g(t)\eta(t)$ exists, possibly equal to $+\infty$ or $-\infty$.

(i) If $\delta > 1$ and (2.5) holds, then (B).

(ii) If $\delta < 1$ and (2.5) holds, then (I).

(iii) If $\delta < 1$ and (2.6) holds, then (U).

(iv) If $\delta > 1$ and (2.6) holds, then (D).

3. EXAMPLES

In this section examples are given which illustrate the applicability of Corollaries 1 and 2. In the process new BT models are introduced.

EXAMPLE 1. A generalization of the gamma and truncated normal densities.

Consider the exponential family of densities of the form

$$f(t) = C(\alpha,\beta,\gamma)\exp\{-\alpha t - \beta t^2 + \gamma \log t\}, \qquad 0 < t < \infty,$$

where the natural parameter space is the union of $\{(\alpha,\beta,\gamma): -\infty<\alpha<\infty,\ \beta > 0,\ \gamma > -1\}$ and $\{(\alpha,\beta,\gamma): \alpha > 0,\ \beta=0,\ \gamma > -1\}$. The special case $\beta=0$ gives the class of gamma densities, which includes the subclass of exponential densities ($\gamma=0$). The special case $\beta > 0$, $\gamma=0$ gives the class of *truncated normal* densities. To apply Corollary 1, note that $k=3$, $M=\infty$, $(\theta_1,\theta_2,\theta_3) = (\alpha,\beta,\gamma)$, and $(U_1(t),U_2(t),U_3(t)) = (-t, -t^2, \log t)$. Therefore $\eta(t) = -\Sigma_{i=1}^{3}\theta_i U_i'(t) = \alpha + 2\beta - \gamma/t$, and $\eta'(t) = 2\beta + \gamma/t^2$.

Case 1. $\beta > 0$. (a) if $\gamma \geq 0$, then $\eta'(t) > 0$ for all $t > 0$, so that by Corollary 1(a), (I) holds. (b) On the other hand, if $\gamma < 0$, then (2.5) holds with $t_0 = (-\gamma/2\beta)^{1/2}$. Since $\lim_{t\downarrow 0} f(t) = \infty$, by Lemma (a,i) we have that (B) holds.

Case 2. $\beta = 0$. (Gamma density.) Here $\eta'(t) = \gamma/t^2$. (a) If $\gamma > 0$, then by Corollary 1(a), (I) holds. (b) If $\gamma < 0$, then by Corollary 1(b), (D) holds. (c) If $\gamma = 0$ (exponential density), then $r(t) = f(t)/R(t) \equiv \alpha$, a constant. These of course are well-known facts. See Barlow and Proschan (1975).

EXAMPLE 2. A generalization of the beta density.

Consider the exponential family of densities of the form

$$f(t) = C(\alpha,\beta)\exp\{\alpha \log(t/M) + \beta \log(1-t/M)\}, \qquad 0 < t < M,$$

where the natural parameter space is $\{(\alpha,\beta):\alpha > -1,\ \beta > -1\}$. In fact, $C(\alpha,\beta) = M^{-1}\Gamma(\alpha+1)\Gamma(\beta+1)/\Gamma(\alpha+\beta+2)$. The case M=1 is the usual beta density. The case $\alpha=\beta=0$ is the uniform density over $(0,M)$. Corollary 1 is applicable here, with k=2, $(\theta_1,\theta_2) = (\alpha,\beta)$, and $(U_1(t),\ U_2(t)) = (\log(t/M),\ \log(1-t/M))$. Consequently, $\eta(t) = \beta/(M-t) - \alpha/t$, and $\eta'(t) = [\beta t^2+\alpha(M-t)^2]/t^2(M-t)^2$.

Case 1. $\alpha<0$, $\beta>0$. Define the quadratic $h(t)$ by $h(t)=\beta t^2+\alpha(M-t)^2 = (\alpha+\beta)t^2-2M\alpha t+\alpha M^2$. Note that $h(0) = \alpha M^2 < 0$ and $h(M)=\beta M^2 > 0$. (a) If $\alpha+\beta = 0$, then $h(t) = -\alpha M(2t-M)$, so that (2.5) holds with $t_0 = M/2$. (b) If $\alpha+\beta>0$, then since h has its minimum at $t = M\alpha/(\alpha+B) < 0$, h is strictly increasing in $(0,M)$, and crosses 0 somewhere in $(0,M)$. Therefore (2.5) holds. (c) If $\alpha+\beta<0$, then since h has its maximum at $t = M\alpha/(\alpha+\beta) > M$, h is strictly increasing in $(0,M)$, and crosses 0 somewhere in $(0,M)$. Therefore (2.5) holds. We conclude that (2.5) holds for any (α,β) satisfying $\alpha<0$, $\beta>0$. Since also $\lim_{t\downarrow 0} f(t) = \infty$, we deduce from Lemma (a,i) that (B) holds.

Case 2. $\alpha>0$, $\beta<0$. An argument analogous to that of Case 1 shows that (U) holds here.

Case 3. $\alpha\geq 0$, $\beta>0$ or $\alpha>0$, $\beta\geq 0$. From Corollary 1(a) it is apparent that (I) holds.

Case 4. $\alpha \leq 0$, $\beta < 0$ or $\alpha < 0$, $\beta \leq 0$. From Corollary 1(b) it is apparent that (D) holds.

Case 5. $\alpha = 0$, $\beta = 0$. Here $r(t) = f(t)/R(t) = 1/(M-t)$, so that (I) holds.

EXAMPLE 3. The lognormal density.

The lognormal density,

$$f(t) = (\sqrt{2\pi}\,\sigma t)^{-1}\exp\{-\tfrac{1}{2}[(\log t - \mu)/\sigma]^2\}, \qquad 0 < t < \infty,\ -\infty < \mu < \infty,\ \sigma > 0,$$

may be written as $f(t) = C(\alpha,\beta)\exp\{-\alpha(\log t)^2 + \beta \log t\}$, where $\alpha = 1/2\sigma^2 > 0$ and $\beta = \mu/\sigma^2 - 1$, $-\infty < \beta < \infty$. Corollary 1 applies with $M = \infty$, $k=2$, $(\theta_1,\theta_2) = (\alpha,\beta)$, and $(U_1(t),U_2(t)) = (-(\log t)^2, \log t)$. Thus $\eta'(t) = (2\alpha + \beta - 2\alpha \log t)/t^2$, and (2.6) holds with $t_0 = \exp\{1 + \beta/2\alpha\}$. Since $\lim_{t\downarrow 0} f(t) = 0$, by Lemma (a,iii) we have that (U) holds. Although this result is well-known (see Mann, Schafer, and Singpurwalla (1974)), our approach is notably swift.

EXAMPLE 4. The inverse Gaussian density.

The inverse Gaussian density,

$$f(t) = (\lambda/2\pi t^3)^{1/2}\exp\{-\lambda(t-\mu)^2/2\mu^2 t\}, \qquad 0 < t < \infty,\ \mu > 0,\ \lambda > 0,$$

has the exponential family form $f(t) = C(\mu,\lambda)\exp\{-(\lambda/2\mu^2)t - (\lambda/2)t^{-1} - (3/2)\log t\}$, which is amenable to Corollary 1 for the case $M = \infty$, $k = 3$, $(\theta_1,\theta_2,\theta_3) = (\lambda/2\mu^2, \lambda/2, -3/2)$, and $(U_1(t),U_2(t),U_3(t)) = (-t, -t^{-1}, \log t)$. Therefore $\eta'(t) = (\lambda - 3t/2)/t^3$, so that (2.6) holds with $t_0 = 2\lambda/3$. Since $\lim_{t\downarrow 0} f(t) = 0$,

by Lemma (a,iii), we have that (U) holds, a fact demonstrated with greater computational effort by Chhikara and Folks (1977).

EXAMPLE 5. A cubic exponential family.

Perhaps the most straightforward way to have (2.5) or (2.6) hold is to make η' linear. This is accomplished in the exponential family case by the class of densities,

$$f(t) = C(\alpha,\beta,\gamma)\exp\{-\alpha t - \beta t^2 - \gamma t^3\}, \qquad t > 0,$$

where the natural parameter space is the union of $\{(\alpha,\beta,\gamma): -\infty<\alpha<\infty, -\infty<\beta<\infty, \gamma>0\}$, $\{(\alpha,\beta,\gamma): -\infty<\alpha<\infty, \beta>0, \gamma=0\}$, and $\{(\alpha,\beta,\gamma):\alpha>0, \beta=0, \gamma=0\}$. In applying Corollary 1, we note that $M=\infty$, $k=3$, $(\theta_1,\theta_2,\theta_3) = (\alpha,\beta,\gamma)$, and $(U_1(t),U_2(t),U_3(t)) = (-t,-t^2,-t^3)$. Therefore $\eta(t) = \alpha + 2\beta t + 3\gamma t^2$, and $\eta'(t) = 2\beta + 6\gamma t$.

Case 1. $\beta\geq 0$, $\gamma>0$ or $\beta>0$, $\gamma\geq 0$. Here by Corollary 1(a), (I) holds.

Case 2. $\beta=\gamma=0$. As noted in Example 1, for this, the exponential density, we have constant failure rate equal to α.

Case 3. $\beta<0$. Then (2.5) holds with $t_0 = -\beta/3\gamma$. Since $\lim_{t\downarrow 0} f(t) = C(\alpha,\beta,\gamma) \neq 0,\infty$, we cannot apply Lemma (a). We therefore consider $\delta = \lim_{t\downarrow 0} g(t)\eta(t) = \alpha/C(\alpha,\beta,\gamma)$. (a) Suppose $\alpha\leq 0$. Then $\delta\leq 0<1$, and by Lemma (b,ii) we have that (I) holds. (b) Suppose $\alpha>0$. Then $\delta=\alpha\int_0^\infty \exp\{-\alpha t - \beta t^2 - \gamma t^3\}dt$, and from Lemma (b,i) and (b,ii) we conclude that (B) holds if $\delta<1$, and (I) holds if $\delta>1$. Further insight is gained by expressing δ as $\delta = \int_0^\infty \exp\{-y + \lambda y^2 - \rho y^3\}dy \equiv h(\lambda,\rho)$, where $\lambda = -\beta/\alpha^2 > 0$

and $\rho = \gamma/\alpha^3 > 0$. Therefore $(\partial/\partial\lambda)h(\lambda,\rho) = \int_0^\infty y^2 \exp\{-y + \lambda y^2 - \rho y^3\} > 0$, so that $h(\lambda,\rho)$ is strictly increasing with λ for fixed ρ. Since $\lim_{\lambda\downarrow 0} h(\lambda,\rho) = \int_0^\infty \exp\{-y - \rho y^3\}dy < \int_0^\infty \exp(-y)dy = 1$, and $\lim_{\lambda\to\infty} h(\lambda,\rho) = \infty$, we conclude from continuity that, holding ρ fixed, for sufficiently small λ we have $\delta < 1$ (I), and for sufficiently large λ we have $\delta > 1$ (B). Similarly, it is seen that $h(\lambda,\rho)$ is strictly decreasing with ρ for fixed λ, and that, holding λ fixed, we have $\delta < 1$ (I) for sufficiently large ρ and $\delta > 1$ (B) for sufficiently small ρ. The case $\delta = 1$ is solvable. Here $\lim_{t\downarrow 0} g'(t) = 0$. However, differentiation of (2.2) gives $\lim_{t\downarrow 0} g''(t) = 2\beta/C(\alpha,\beta,\gamma) < 0$. Thus g is initially decreasing, and (I) holds.

EXAMPLE 6. Gamma mixtures with common scale parameter.

Consider the mixture of densities, $f(t) = p\,f_1(t) + q\,f_2(t)$, $0 < p < 1$, $q = 1-p$, where f_j, $j = 1,2$, is gamma distributed with shape parameter γ_j and scale parameter α, i.e.,

$$f_j(t) = \left[\alpha^{\gamma_j}/\Gamma(\gamma_j)\right] t^{\gamma_j - 1} e^{-\alpha t}, \qquad t > 0,\ \alpha > 0,\ \gamma_j > 0.$$

For integer valued γ_j, f_j is the density of the time until the occurence of the γ_j-th event of a Poisson process with parameter α. If an item fails upon the accumulation of precisely γ_j Poisson (α) occurrences (say shocks), then f_j is the appropriate density function for the item's lifetime. If a population consists of a mixture having proportion p of items failing upon γ_1 Poisson (α) shocks and proportion $q = 1-p$ of items failing upon γ_2 Poisson (α) shocks, then f is the appropriate density. We generalize

here to all positive, not necessarily integer valued γ_j. We further assume without loss of generality that $\gamma_1 > \gamma_2$.

We are motivated to look for possible BT mixtures by considering the following representation of r(t). Suppose $\gamma_2 < 1$ and $\gamma_1 > 1$, i.e., f_1 is IFR and f_2 is DFR. We have

$$r(t) = [p\,f_1(t) + q\,f_2(t)]/[p\,R_1(t) + q\,R_2(t)] = h_1(t)r_1(t) + h_2(t)r_2(t),$$

where R_j is the reliability and r_j the failure rate for f_j, $h_1(t) = [1+c\,h(t)]^{-1}$, $h_2(t) = 1-h_1(t)$, and $h(t) = R_2(t)/R_1(t)$, with $c = q/p$. Thus r(t) is a weighted average of the component failure rates, where the weights vary with t. In fact, it is easily shown that h(t) is strictly decreasing, ranging from 1 at t=0 to 0 at $t = \infty$. Thus the weight function $h_1(t)$ increases from p to 1, whereas $h_2(t)$ decreased from q to 0. Since $\lim_{t\downarrow 0} r_2(t) = \infty$, but $\lim_{t\downarrow 0} r_1(t) = 0$, r(t) has an initial DFR character. Since $\lim_{t\to\infty} r_j(t) = \alpha$ and $\lim_{t\to\infty} h_1(t) = 1$, r(t) has an ultimate IFR character. We shall show by Corollary 2 that indeed f is BT for this situation.

To use Corollary 2, we write f_j as $f_j(t) = \left[\alpha^{\gamma_j}/\Gamma(\gamma_j)\right]\exp\{-\alpha t + (\gamma_j - 1)\log t\}$, $t > 0$, which gives $M = \infty$, k=2, $(\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}) = (\alpha, \alpha, \gamma_1 - 1, \gamma_2 - 1)$, and $(U_1(t), U_2(t)) = (-t, \log t)$. From (2.11), we obtain $\xi(t) = 1 + b\,t^a$, where $a \equiv \gamma_1 - \gamma_2 > 0$ and $b \equiv (\alpha^a/c)\Gamma(\gamma_2)/\Gamma(\gamma_1) > 0$. From (2.12), we obtain $\eta(t) = \alpha - \epsilon/t + a/t(1+b\,t^a)$, where $\epsilon \equiv \gamma_1 - 1$. Consequently, $\eta'(t) = [t^2(1+b\,t^a)^2]^{-1}[\epsilon b^2 t^{2a} + [2\epsilon - a(a+1)]b\,t^a + \epsilon - a]$. In investigating η' via Corollary 2 it suffices to consider the behavior of the quadratic $h(w) \equiv \epsilon w^2 + [2\epsilon - a(a+1)]w + \epsilon - a$, where $w = w(t) \equiv b\,t^a$.

Case 1. $\gamma_1 \leq 1$. Here $\epsilon \leq 0$. If $\epsilon = 0$, then $h(w) = -a(a+1)w - a < 0$ for all $w > 0$. If $\epsilon < 0$, then since $h(0) = \epsilon - a < 0$ and $h'(w) < 0$ for all $w > 0$, we have $h(w) < 0$ for all $w > 0$. Therefore $\eta'(t) < 0$ for all $t > 0$, which implies by Corollary 2(b) that (D) holds. This fact is consistent with the general result (see Barlow and Proschan (1975)) that mixtures of DFR distributions are DFR.

Case 2. $\gamma_1 > 1$, $\gamma_2 < 1$. Here $h(0) = \gamma_2 - 1 < 0$, which implies that for some $w_0 > 0$, we have $h(w) < 0$ for all $w < w_0$, $h(w_0) = 0$, and $h(w) > 0$ for all $w > w_0$. Therefore (2.5) holds with $t_0 = (w_0/b)^{1/a}$. Since $\lim_{t \downarrow 0} f(t) = \infty$, it follows from Lemma (a,i) that (B) holds. To summarize, if one density is strictly IFR and the other is strictly DFR, then the mixture is BT.

Case 3. $\gamma_1 > 1$, $\gamma_2 = 1$. Here f_2 is the exponential density, $h(0) = 0$, and $h'(0) = a(1-a)$. If $a < 1$, then $h'(0) > 0$, so that $h(w) > 0$ for all $w > 0$. If $a = 1$, then $h'(0) = 0$. However, $h''(w) = 2\epsilon$ implies that $h(w) > 0$ for all $w > 0$. Therefore, if $a \leq 1$, we have $\eta'(t) > 0$ for all $t > 0$, so that by Corollary 2(a), (I) holds. On the other hand, if $a > 1$, then $h'(0) < 0$, which implies that (2.5) holds. Since $\lim_{t \downarrow 0} f(t) = q\alpha \neq 0, \infty$, we cannot use Lemma (a). However, since $\delta = 1/q > 1$, Lemma (b,i) shows that (B) holds. To summarize, if f_1 is strictly IFR and f_2 is exponential, then the mixture is IFR if $\gamma_1 \leq 2$ and BT if $\gamma_1 > 2$.

Case 4. $\gamma_1 > 1$, $\gamma_2 > 1$. Here both densities are strictly IFR, and $h(0) > 0$. Since $h'(w) = 2\epsilon w + 2\epsilon - a(a+1)$ is an increasing function of w, it follows that if $h'(0) \geq 0$, then $h(w) > 0$ for all $w > 0$. Now $h'(0) \geq 0$ if and only if $d \geq a(a-1)/2$, where $d \equiv \epsilon - a$. If $a \leq 1$, this inequality is

satisfied for all d. Suppose $a > 1$ and $d < a(a-1)/2$, so that $h'(0) < 0$. Therefore $h(w) > 0$ for all $w > 0$ if and only if $h(w) = 0$ has no real roots. But no real roots exist if and only if $[2\varepsilon - a(a+1)]^2 - 4\varepsilon(\varepsilon - a) < 0$, i.e., $(a-1)^2 < 4d$. Therefore, by Corollary 2(a), we have that (I) holds for the following cases: $a \leq 1$; $a > 1$, $d \geq a(a-1)/2$; and $a > 1$, $(a-1)^2/4 < d < a(a-1)/2$. For the case $a > 1$, $d = (a-1)^2/4$, we have for $w_0 = 2a/(a+1)$ that $h(w) > 0$ for all positive $w \neq w_0$, with $h(w_0) = 0$. It follows that for $t_0 = (w_0/b)^{1/a}$, we have $\eta'(t) > 0$ for all positive $t \neq t_0$, with $\eta'(t_0) = 0$. From (2.4) it then follows that (I) holds. For the remaining case $a > 1$, $d < (a-1)^2/4$, it is apparent that η' is initially positive, then negative, and ultimately positive. The results of Section 2 are therefore not applicable. However, we may rule out f as BT or DFR, since $\lim_{t \downarrow 0} f(t) = 0$, which implies $g(t)$ is initially decreasing. Further, since η is ultimately increasing, we deduce from (2.4) that g is ultimately decreasing. This rules out UBT. A reasonable conjecture seems to be that f is IFR in this case as well.

EXAMPLE 7. General gamma mixtures.

We now generalize the mixture of Example 6 to the case of unequal scale parameters, i.e., for $j=1,2$,

$$f_j(t) = \left[\alpha_j^{\gamma_j}/\Gamma(\gamma_j)\right] t^{\gamma_j - 1} e^{-\alpha_j t}, \qquad t > 0,\ \alpha_j > 0,\ \gamma_j > 0.$$

The analysis for this situation is more complicated, and we restrict consideration to the case $0 < \gamma_2 < 1 < \gamma_1 < \infty$, which is BT in the $\alpha_1 = \alpha_2$

context. We use the same definitions and assignments given in Example 6, except now $\theta_{11} = \alpha_1$, $\theta_{12} = \alpha_2$, and $b = \left(\alpha_1^{\gamma_1}/c\,\alpha_2^{\gamma_2}\right)\Gamma(\gamma_2)/\Gamma(\gamma_1) > 0$. Further, we define $\lambda \equiv \alpha_2 - \alpha_1$. Then $\eta(t) = \alpha_1 - \epsilon/t + (a+\lambda t)/t(1+b\,e^{\lambda t}t^a)$, so that $\eta'(t) = [t^2(1+w)^2]^{-1}[\epsilon(1+w)^2 - (a+\lambda t)^2 w - a(1+w)]$, where $w = w(t) \equiv b\,e^{\lambda t}t^a$. Use of Corollary 2 involves investigation of $h(t) \equiv \epsilon(1+w)^2 - (a+\lambda t)^2 w - a(1+w)$.

Case 1. $\lambda < 0$. Here $w(0) = w(\infty) = 0$, and $h(0) = h(\infty) = \epsilon - a < 0$. Now η ultimately decreasing rules out IFR and BT, and $\lim_{t\downarrow 0} f(t) = \infty$ rules out IFR and UBT. DFR remains a possibility. In fact, $h(t) < 0$ for all $t > 0$ would imply (D) from Corollary 2(b). However, investigation of h at extrema shows that $h(t^*) = [1+w(t^*)]\{\epsilon[1+w(t^*)]-a\}$, where $t^* \equiv -a/\lambda$. Thus for suitable choices of (a,b,ϵ), $h(t^*) > 0$ and Corollary 2 does not apply.

Case 2. $\lambda > 0$. Here $w(0) = 0$ and $w(\infty) = \infty$, so that $h(0) < 0$ and $h(\infty) > 0$. It is convenient to express $h(t)$ as $h(t) = a\,s(m)$, where $m \equiv \lambda t/a$. Then $s(m) = d(1+z)^2 - a(m+1)^2 z - z - 1$, where $d \equiv \epsilon/a$, $(0 < d < 1)$, $z = z(m) \equiv \mu\,e^{am}m^a$, and $\mu \equiv b(a/\lambda)^a$. Since $s(0) < 0$ and $s(\infty) > 0$, there exists at least one positive root to $s(m) = 0$. Suppose m_0 is such a root. In order to show that (2.5) holds, it suffices to show that $s'(m_0) > 0$. For any m, $s'(m) = a[(m+1)/m][2d(1+z)z - 2mz - a(m+1)^2 z - z]$. Since $s(m_0) = 0$, we have $2dz_0^2 = 2a(m_0+1)^2 z_0 + 2z_0 + 2 - 2d - 4dz_0$ (where $z_0 \equiv z(m_0)$), so that $s'(m_0) = a[(m_0+1)/m_0]\{z_0[a(m_0+1)^2 - 2d - 2m_0 + 1] + 2(1-d)\}$. Consider the quadratic $v(m) \equiv a(m+1)^2 - 2d - 2m + 1$. (a) Suppose $a \geq 1$. Then $v(0) = a - 2d + 1 \geq 2(1-d) > 0$, and $v(m)$ is strictly increasing for positive m. Therefore $v(m_0) > 0$, which implies $s'(m_0) > 0$, and (2.5) holds. From Lemma (a,i) we conclude that (B) holds. (b) Now suppose $a < 1$. In this situation, unlike the $a \geq 1$ case, the number of positive roots to

$s(m) = 0$ may exceed 1. This is because $v'(0) = 2(a-1) < 0$ allows for a root m_0 satisfying $s'(m_0) < 0$. As an example, consider $a = 3/16$, $\epsilon = 3/32$, $\mu = 2\exp(-3/16)$. Then $s(1) = 0$ but $s'(1) = -9/16$. In fact in this example, $s(m) = 0$ has 3 positive roots so that η' is initially negative, then positive, then negative, and finally positive. We may summarize the case $a < 1$ by stating that (B) holds if there is precisely one positive root to $s(m) = 0$. If the number of such roots exceeds 1, our general results are not applicable, although it is easy to see that IFR, DFR, and UBT are ruled out. A seemingly reasonable conjecture is that f is BT here also.

We conclude this example by considering the special case $\gamma_2 = 1$ (exponential density), with $a \geq 1$ and $\lambda > 0$. Recall from Example 6 that for $\lambda = 0$, f is BT if $a > 1$ and IFR if $a = 1$. We shall discover that in the present case, f is BT if $a > 1$ and either IFR or BT if $a = 1$. Note that for $\gamma_2 = 1$, we have $d = 1$. Therefore $s(0) = 0$ and $s(\infty) = \infty$. The existence of a positive root to $s(m) = 0$ is not assured. If one exists, say $m = m_0$, the argument given above for $a \geq 1$ still applies to show that $v(m_0) > 0$ and $v'(m_0) > 0$, and thus that (2.5) holds. On the other hand, if no positive root exists, we conclude that $h(t) > 0$ for all $t > 0$, so that (I) holds. To show existence of a positive root, it suffices to show s is initially decreasing. Since $z' = z'(m) = a[(m+1)/m]z$, we may express $s'(m)$ as $s'(m) = z'\psi$, where $\psi = \psi(m) \equiv 1+2z - 2m - a(m+1)^2$. Therefore $s'(0) = a(1-a)\lim_{m\downarrow 0}(z/m) = 0$, since $\lim_{m\downarrow 0}(z/m) = \mu$ if $a = 1$, and $= 0$ if $a > 1$. Consider therefore $s''(0)$. We have $s''(m) = z'\psi' + z''\psi$,

with $z'' = z''(m) = a[a+2a/m + (a-1)/m^2]z$. (a) Suppose $a = 1$. Then $z'(0) = \mu$, $\psi'(0) = 2\mu-4$, $z''(0) = 2\mu$, and $\psi(0) = 0$, so that $s''(0) = 2\mu(\mu-2) < 0$ only if $\mu < 2$. It can be shown that $s'''(0) = 12$ for $\mu = 2$. We conclude that for $a = 1$, s is initially decreasing only if $\mu < 2$. (b) Suppose $a > 1$. Then $z'(0) = 0$, $\psi'(0) = -2(a+1)$, $z''(0) = a(a-1)\lim_{m\downarrow 0}(z/m^2)$, and $\psi(0) = 1-a$. Since $\lim_{m\downarrow 0}(z/m^2) = \infty$ if $1 < a < 2$, $= \mu$ if $a = 2$, and $= 0$ if $a > 2$, we have that $s''(0) = -\infty < 0$ if $1 < a < 2$, $s''(0) = -2\mu < 0$ if $a = 2$, and $s''(0) = 0$ if $a > 2$. By taking higher derivatives it can be shown that for $a > n-1$, $s^{(i)}(0) = 0$ for $i \leq n-1$, and $s^{(n)}(0) = (1-a)z^{(n)}(0)$, with $z^{(n)}(0) = a(a-1)\cdots(a-n+1)\lim_{m\downarrow 0}(z/m^n)$. Consequently, for all $a > 1$, we have s initially decreasing. In all instances where s initially decreases (implying (2.5) holds), we need to consider the Lemma. Since $\gamma_2 = 1$, we have $\lim_{t\downarrow 0} f(t) = q\,\alpha_2$, and Lemma (a) does not apply. However, for $a > 1$, $\lim_{t\downarrow 0}\eta(t) = \alpha_1 + \lambda = \alpha_2$, so that $\delta = 1/q > 1$, and Lemma (b,i) concludes (B). For $a = 1$ with $\mu < 2$, $\lim_{t\downarrow 0}\eta(t) = \alpha_1 + \lambda - b = \alpha_2 - b$. Therefore $\delta = 1/q - p(\alpha_1/q\,\alpha_2)^2$, which may be less than or greater than 1, depending on (p,α_1,α_2). The following summarizes our results in the exponential case ($\gamma_2 = 1$), where $\lambda > 0$ and $a \geq 1$. If $a > 1$, or if $a = 1$, $\mu < 2$, and $1/q - p(\alpha_1/q\,\alpha_2)^2 > 1$, then (B) holds. If $a = 1$ and $\mu \geq 2$, or if $a = 1$, $\mu < 2$, and $1/q - p(\alpha_1/q\,\alpha_2)^2 < 1$, then (I) holds. Since the case $\lambda = 0$ implies $\mu = \infty$, these results are consistent with the $\lambda = 0$ counterparts in Example 6.

EXAMPLE 8. Weibull mixtures.

In Example 6, we showed that the mixture is BT for a strictly IFR gamma density and a strictly DFR gamma density with the same scale parameter.

Here we show that the corresponding mixture based on Weibull densities is not BT. This disputes a contention of Kao (1959).

Consider the mixture $f(t) = p\,f_1(t) + q\,f_2(t)$, where

$$f_j(t) = \alpha\beta_j(\alpha t)^{\beta_j - 1}\exp\{-(\alpha t)^{\beta_j}\}, \qquad t > 0,\ \alpha > 0,\ \beta_j > 0.$$

The failure rates have the concise form $r_j(t) = \alpha\beta_j(\alpha t)^{\beta_j - 1}$. Now $r(t) = h_1(t)r_1(t) + h_2(t)r_2(t)$, where $h_1(t) = [1 + c\,h(t)]^{-1}$ and $h_2(t) = 1 - h_1(t)$, with $c = q/p$ and $h(t) = R_2(t)/R_1(t) = \exp\{(\alpha t)^{\beta_1} - (\alpha t)^{\beta_2}\}$. By assuming $0 < \beta_2 < 1 < \beta_1 < \infty$, we have f_1 IFR and f_2 DFR. But then $\lim_{t \downarrow 0} r(t) = \infty$ and $\lim_{t \to \infty} r(t) = 0$. Therefore f cannot be BT. In fact, it is seen that $r(t)$ initially decreases, then increases, and then ultimately decreases for the given mixture. We may generalize to unequal scale parameters, $\alpha_1 \neq \alpha_2$. Again $\lim_{t \downarrow 0} r(t) = \infty$ and $\lim_{t \to \infty} r(t) = 0$, which rules out BT. The same conclusion holds when f_2 is Weibull with a positive valued location parameter. Therefore Kao's contention that such Weibull mixtures are BT is incorrect.

4. STATISTICAL INFERENCE

The models in Examples 1 through 5 have the exponential family of densities form. As a result, statistical inference is simplified by a reduction to sufficient statistics, and optimum hypothesis tests and confidence region procedures can be obtained. See Lehmann(1959). We consider in this section certain point estimation techniques, particularly for the model of Example 1.

THE GENERALIZED GAMMA-TRUNCATED NORMAL DENSITY OF EXAMPLE 1.

The parametrization given in Section 3 obscures the role of α in the $\beta > 0$ situation. Since $|\alpha|$ is a scale parameter unless $\alpha = 0$, it seems natural to break the overall model into the following four classes, which effectively delineate these roles of α. $1:(\alpha > 0,\ \beta > 0,\ \gamma > -1)$. $2:(\alpha < 0,\ \beta > 0,\ \gamma > -1)$. $3:(\alpha = 0,\ \beta > 0,\ \gamma > -1)$. $4:(\alpha > 0,\ \beta = 0,\ \gamma > -1)$. For convenience in estimation, we re-parametrize slightly.

Class 1: $f(t) = [\alpha^{\rho}/\Gamma(\theta,\rho)]\exp\{-\alpha t - \theta(\alpha t)^2\}t^{\rho-1}$, $\quad \alpha > 0,\ \theta > 0,\ \rho > 0.$

$\Gamma(\theta,\rho)$ is defined by $\Gamma(\theta,\rho) = \int_0^{\infty}\exp\{-y - \theta y^2\}y^{\rho-1}dy$.

Class 2: $f(t) = [\sigma^{\rho}/\wedge(\theta,\rho)]\exp\{\sigma t - \theta(\sigma t)^2\}t^{\rho-1}$, $\quad \sigma > 0,\ \theta > 0,\ \rho > 0.$

$\wedge(\theta,\rho)$ is defined by $\wedge(\theta,\rho) = \int_0^{\infty}\exp\{y - \theta y^2\}y^{\rho-1}dy$.

Class 3: $f(t) = [\lambda^{\rho}/H(\rho)]\exp\{-(\lambda t)^2\}t^{\rho-1}$, $\quad \lambda > 0,\ \rho > 0.$

$H(\rho)$ is defined by $H(\rho) = \int_0^{\infty}\exp\{-y^2\}y^{\rho-1}dy$.

Class 4: $f(t) = [\alpha^{\rho}/\Gamma(\rho)]e^{-\alpha t}t^{\rho-1}$, $\quad \alpha > 0,\ \rho > 0.$

Note that in Classes 1, 2, and 3, f is BT if $\rho < 1$, IFR if $\rho \geq 1$, and truncated normal if $\rho = 1$. Class 4 is the class of gamma densities, where f is DFR if $\rho < 1$, IFR if $\rho > 1$, and exponential if $\rho = 1$.

Maximum likelihood estimation of the parameters is tractable for this model. In practice, the statistician would form a collection of possible lifetime models by choosing some or all of the four classes. Typical choices might be Classes $\{1,2,3,4\}$, or $\{1,2\}$, or $\{1,4\}$, or $\{2,4\}$, or $\{1,2\}$. For each class included, the MLE of the corresponding parameters

would be computed, as well as the value of the likelihood at this point. The MLE of (α,β,γ) would then be the value of (α,β,γ) corresponding to the MLE of the parameters in the class with the largest maximum likelihood. For example, suppose we wish to estimate (α,β,γ), assuming the collection of possible lifetime densities is the set of Classes $\{1,4\}$. Maximum likelihood estimation for the individual classes yields the MLEs, say $(\alpha_1,\theta_1,\rho_1)$ for Class 1 and (α_4,ρ_4) for Class 4. If the likelihood evaluated for Class 1 at $(\alpha_1,\theta_1,\rho_1)$ exceeds the likelihood evaluated for Class 4 at (α_4,ρ_4), then the MLE of (α,β,γ) is $(\alpha_1,\theta_1\alpha_1^2,\rho_1-1)$. If the likelihood (4) at (α_4,ρ_4) exceeds the likelihood (1) at $(\alpha_1,\theta_1,\rho_1)$, then the MLE of (α,β,γ) is $(\alpha_4,0,\rho_4-1)$.

We present now the individual MLEs for the four classes. Assume a random sample of failure times is taken, with observed values $t_1,\ldots,t_n$. Computation of MLEs is simplified by the exponential family form. In general, if

$$f(t) = C(\underset{\sim}{\theta})h(t)\exp\{\Sigma_{i=1}^{k}\theta_i U_i(t)\}, \qquad 0<t<M, \tag{4.1}$$

then it is easily seen that any solution to the set of equations, $(\partial/\partial\theta_i)\log L = 0$, $i=1,\ldots,k$, (where the likelihood $L \equiv \Pi_{j=1}^{n} f(t_j)$), is a solution to the set of equations,

$$E\,U_i(T) = S_i(\underset{\sim}{t}), \qquad i=1,\ldots,k, \tag{4.2}$$

where T denotes the time-to-failure random variable, and $(S_1(\underset{\sim}{t}),\ldots,S_k(\underset{\sim}{t}))$ is the sufficient statistic given by $S_i(\underset{\sim}{t}) = \frac{1}{n}\Sigma_{j=1}^{n}U_i(t_j)$.

Class 1. Here $(S_1(\underset{\sim}{t}), S_2(\underset{\sim}{t}), S_3(\underset{\sim}{t})) = (\frac{1}{n}\Sigma t_j, \frac{1}{n}\Sigma t_j^2, \frac{1}{n}\Sigma \log t_j)$, and $(E\,U_1(T), E\,U_2(T), E\,U_3(T)) = (E\,T, E\,T^2, E \log T)$. Now $E\,T = \int_0^\infty [\Gamma(\theta,\rho+1)/\alpha\Gamma(\theta,\rho)][\alpha^{\rho+1}/\Gamma(\theta,\rho+1)]\exp\{-\alpha t-\theta(\alpha t)^2\}t^\rho dt = \Gamma(\theta,\rho+1)/\alpha\Gamma(\theta,\rho)$. Similarly, $E\,T^2 = \Gamma(\theta,\rho+2)/\alpha^2\Gamma(\theta,\rho)$. In a like manner, $E \log T = \psi(\theta,\rho) - \log\alpha$, where $\psi(\theta,\rho) \equiv [\Gamma(\theta,\rho)]^{-1}\int_0^\infty \exp\{-y-\theta y^2\}y^{\rho-1}\log y\,dy = (\partial/\partial\rho)\log\Gamma(\theta,\rho)$. We may solve the system (4.2) to obtain the MLE $(\alpha_1,\theta_1,\rho_1)$. A computational version of the system is the set of equations $S_1^2/S_2 = \Gamma^2(\theta_1,\rho_1+1)/\Gamma(\theta_1,\rho_1)\Gamma(\theta_1,\rho_1+2)$, $S_3 - \log S_1 = \psi(\theta_1,\rho_1) - \log[\Gamma(\theta_1,\rho_1+1)/\Gamma(\theta_1,\rho_1)]$, and $\alpha_1 = \Gamma(\theta_1,\rho_1+1)/S_1\Gamma(\theta_1,\rho_1)$. The first two equations may be solved simultaneously (by computer) to yield (θ_1,ρ_1). The third equation then provides α_1.

Class 2. Here the $S_i(\underset{\sim}{t})$ and $U_i(T)$ are identical to those in Class 1. It is easily seen that $E\,T = \Lambda(\theta,\rho+1)/\sigma\Lambda(\theta,\rho)$, $E\,T^2 = \Lambda(\theta,\rho+2)/\sigma^2\Lambda(\theta,\rho)$, and $E\log T = \zeta(\theta,\rho) - \log\sigma$, where $\zeta(\theta,\rho) \equiv [\Lambda(\theta,\rho)]^{-1}\int_0^\infty \exp\{y-\theta y^2\}y^{\rho-1}\log y\,dy = (\partial/\partial\rho)\log\Lambda(\theta,\rho)$. The resultant computational version of the system (4.2) which yields the MLE $(\sigma_2,\theta_2,\rho_2)$ is the set of equations $S_1^2/S_2 = \Lambda^2(\theta_2,\rho_2+1)//\Lambda(\theta_2,\rho_2)\Lambda(\theta_2,\rho_2+2)$, $S_3 - \log S_1 = \zeta(\theta_2,\rho_2) - \log[\Lambda(\theta_2,\rho_2+1)/\Lambda(\theta_2,\rho_2)]$ and $\sigma_2 = \Lambda(\theta_2,\rho_2+1)/S_1\Lambda(\theta_2,\rho_2)$.

Class 3. Here $(S_1(\underset{\sim}{t}), S_2(\underset{\sim}{t})) = (\frac{1}{n}\Sigma t_j^2, \frac{1}{n}\Sigma \log t_j)$, and $(E\,U_1(T), E\,U_2(T)) = (ET^2, E\log T)$. Then $E\,T^2 = H(\rho+2)/\lambda^2 H(\rho)$, and $E\log T = W(\rho) - \log\lambda$, where $W(\rho) \equiv [H(\rho)]^{-1}\int_0^\infty \exp\{-y^2\}y^{\rho-1}\log y\,dy = (\partial/\partial\rho)\log H(\rho)$. A computational version of (4.2) yielding the MLE (λ_3,ρ_3) is the set of equations $S_3 - \frac{1}{2}\log S_2 = W(\rho_3) - \frac{1}{2}\log[H(\rho_3+2)/H(\rho_3)]$, and $\lambda_3^2 = H(\rho_3+2)/S_2H(\rho_3)$.

Class 4. Here $(S_1(\underset{\sim}{t}), S_2(\underset{\sim}{t})) = (\frac{1}{n}\Sigma t_j, \frac{1}{n}\Sigma \log t_j)$, and $(E\,U_1(T), E\,U_2(T)) = (E\,T, E \log T)$. Since $E\,T = \rho/\alpha$, and $E \log T = \psi(\rho) - \log\alpha$, where $\psi(\rho) = (\partial/\partial\rho)\log\Gamma(\rho)$, we obtain the well-known result that the MLE (α_4, ρ_4) is determined by $S_3 - \log S_1 = \psi(\rho_4) - \log \rho_4$, and $\alpha_4 = \rho_4/S_1$.

Large sample confidence regions for the parameters of each individual class can be obtained from asymptotic normal theory. Let $\hat{\underset{\sim}{\theta}}$ denote the MLE of $\underset{\sim}{\theta}$, based on a random sample of size n taken from (4.1). Under suitable regularity conditions (see Zacks (1971)), the limiting distribution of $\sqrt{n}(\hat{\underset{\sim}{\theta}} - \underset{\sim}{\theta})$ is k-dimensional multivariate normal with mean vector $\underset{\sim}{0}$ and covariance matrix $\underset{\sim}{I}^{-1}(\underset{\sim}{\theta})$, where $\underset{\sim}{I}(\underset{\sim}{\theta})_{ij} = -E(\partial^2/\partial\theta_i\partial\theta_j)\log f$. It is straightforward to show that for Class 1,

$$I(\alpha,\theta,\rho) = \begin{bmatrix} \frac{\rho}{\alpha^2} + \frac{2\theta}{\alpha^2}\frac{\Gamma(\theta,\rho+2)}{\Gamma(\theta,\rho)} & \frac{2}{\alpha}\frac{\Gamma(\theta,\rho+2)}{\Gamma(\theta,\rho)} & -\frac{1}{\alpha} \\ \frac{2}{\alpha}\frac{\Gamma(\theta,\rho+2)}{\Gamma(\theta,\rho)} & \frac{\Gamma(\theta,\rho+4)}{\Gamma(\theta,\rho)} - \left[\frac{\Gamma(\theta,\rho+2)}{\Gamma(\theta,\rho)}\right]^2 & \frac{\Gamma(\theta,\rho+2)}{\Gamma(\theta,\rho)}\left[\psi(\theta,\rho)-\psi(\theta,\rho+2)\right] \\ -\frac{1}{\alpha} & \frac{\Gamma(\theta,\rho+2)}{\Gamma(\theta,\rho)}\left[\psi(\theta,\rho)-\psi(\theta,\rho+2)\right] & \frac{\partial^2}{\partial\rho^2}\log\Gamma(\theta,\rho) \end{bmatrix} \tag{4.3}$$

For Class 2, $I(\sigma,\theta,\rho)$ is identical to (4.3) if we substitute σ for α, and the function Λ for Γ. For Class 3,

$$I(\lambda,\rho) = \begin{bmatrix} \frac{\rho}{\lambda^2} + \frac{2}{\lambda^2} \frac{H(\rho+2)}{H(\rho)} & -\frac{1}{\lambda} \\ -\frac{1}{\lambda} & \frac{\partial^2}{\partial\rho^2} \log H(\rho) \end{bmatrix} .$$

As is well-known, for Class 4,

$$I(\alpha,\rho) = \begin{bmatrix} \frac{\rho}{\alpha^2} & -\frac{1}{\alpha} \\ -\frac{1}{\alpha} & \frac{\partial^2}{\partial\rho^2} \log \Gamma(\rho) \end{bmatrix} .$$

THE OTHER EXAMPLES. We consider briefly questions of inference for Examples 2 through 7 from Section 3.

In Example 2, if T denotes the time-to-failure random variable, then T/M has the usual two parameter beta density over $(0,1)$. Assuming M to be known, we may employ standard inference procedures for (α,β). See Johnson and Kotz (1970). Available procedures likewise handle the lognormal (see Mann, Schafer, and Singpurwalla (1974)) and inverse Gaussian densities (see Chhikara and Folks (1977)).

For the cubic exponential family case of Example 5, we see from (4.2) that the MLE of (α,β,γ) is the method of moments estimator. Computation would require the use of a computer, due to the complicated nature of $C(\alpha,\beta,\gamma)$.

For Examples 6 and 7, the lack of non-trivial sufficient statistics makes the method of maximum likelihood forbidding. Method of moments estimators are, however, computable. The i-th moment, μ_i, of f is $\mu_i = p\mu_{1i} + q\mu_{2i}$, where $\mu_{ji} = \Gamma(\gamma_j+i)/\alpha_j^i\Gamma(\gamma_j)$ is the i-th moment of f_j, j=1,2. The system of equations, $\mu_i = \frac{1}{n}\Sigma_{j=1}^n t_j^i$, i=1,...,k, yields the desired estimates. In Example 6, we use k=4 to estimate $(\alpha,\gamma_1,\gamma_2,p)$, and in Example 7, we use k=5 for $(\alpha_1,\alpha_2,\gamma_1,\gamma_2,p)$. Solution, of course, requires use of a computer.

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